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AN AXIOMATIC CHARACTERIZATION OF THE DIMENSION OF SUBSETS OF EUCLIDEAN SPACES

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Menger's axioms on dimension functions are not adequate to determine the dimension on the class of all subspaces of Euclidean spaces. In this paper, the following is shown: A decomposition axiom together with Menger's axioms characterize the dimension on the class of all subspaces of Euclidean spaces, and the axioms are independent.

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1. Axioms

Let $\mathcal{Q}$ be a class of spaces, $\mathcal{E}$ the class of all subspaces of Euclidean spaces and $\mathcal{S}$ the class of all separable metric spaces. In 1929, Menger studied real-valued set functions d defined on the class $\mathcal{Q}$ satisfying the following conditions (A1)-(A5):

(A1) monotone: if $X \in \mathcal{Q}$ and $X' \subset X$, then $d(X') \leq d(X)$,

(A2) F_σ-constant: if X ($\in \mathcal{Q}$) is the countable sum of closed sets X_i, then $d(X) \leq \max_i d(X_i)$,

(A3) topological: if X ($\in \mathcal{Q}$) and X' are homeomorphic, then $d(X) = d(X')$,

(A4) compactifiable: each X ($\in \mathcal{Q}$) is homeomorphic to a subset of a compact space A ($\in \mathcal{Q}$) such that $d(X) = d(A)$,

(A5) normed: $d(\emptyset) = -1$, $d(I^n) = n$ $(n = 0, 1, 2, \ldots)$ for the n-dimensional interval I^n.

Menger raised a problem: "In the case $\mathcal{Q} = \mathcal{E}$, does d coincide with the dimension dim ?", and he proved that if $\mathcal{Q}$ is the class of all subspaces of the 2-dimensional Euclidean space, then d coincides with dim [7].

In 1968, Shvedov showed that the conditions (A1)-(A5) are not adequate to determine the dimension in the case $\mathcal{Q} = \mathcal{E}$ (this result is introduced in [4]).

In order to characterize dimension, what kind of conditions should be added besides the above conditions (A1)-(A5)? Many proposals regarding characterization of dimension axiomatically are already known (for instance, [1, 3, 6, 8-13] (for metric spaces)). However, as far as we know, there has been no proposal complementing the idea of Menger. In the present paper, we prove that the condition

(A6) if $d(X) < n$ (where n is a positive integer), then there exist n sets X_i such that $X = \bigcup_{i=1}^{n} X_i$ and $d(X_i) \leq 0$ $(i = 1, 2, \ldots, n)$
together with the conditions (A1)-(A5) characterize dimension in the case $\mathcal{Q} = \mathcal{E}$, and that the conditions (A1)-(A6) are independent.

Throughout this paper, spaces are subspaces of Euclidean spaces unless otherwise specified.

In cases $\mathcal{Q} = \mathcal{E}$ or $\mathcal{Q} = \mathcal{S}$, as dim is a function satisfying the conditions (A1)-(A6), it is obvious that the conditions (A1)-(A6) are consistent and that in order that a set function d defined on $\mathcal{Q}$ coincides with dim it is necessary that d satisfies the conditions (A1)-(A6).

2. Definitions

Throughout this paper, I is the unit closed interval of real numbers, i.e., $I = \{x \mid x$ is a real number, $0 \leq x \leq 1\}$, R^m is the m-dimensional Euclidean space, and I^m is the unit cube in R^m. The set of all points $x \in I^m$ such that at most n of the coordinates of x are irrational is denoted by I_n^m. Each point x in R^m is represented by $(x^{(1)}, x^{(2)}, \ldots, x^{(m)})$, where $x^{(i)}$ is the ith coordinate of x. A *parallelotope* in R^m is a set having the form $\{x \in R^m \mid a_i \leq x^{(i)} \leq b_i \ (i = 1, 2, \ldots, m)\}$, where a_i and b_i are real numbers with $a_i < b_i$ $(i = 1, 2, \ldots, m)$, and a *half open parallelotope* in R^m is a set having the form $\{x \in R^m \mid a_i \leq x^{(i)} \leq b_i \ (i = i[1], i[2], \ldots, i[n]),\ a_i < x^{(i)} < b_i \ (i \neq i[1], i[2], \ldots, i[n])\}$, where $\{i[1], i[2], \ldots, i[n]\}$ is a proper subset of $\{1, 2, \ldots, m\}$. The closure of a half open parallelotope is a parallelotope. The definition of a *face* of a parallelotope should be evident. Each face of the closure of a half open parallelotope P is called a *face* of P. In R^m, an *n-parallelotope* $(n < m)$ is a set having the form $\{x \in R^m \mid a_i \leq x^{(i)} \leq b_i \ (i = i[1], i[2], \ldots, i[n]),\ x^{(i)} = a_i \ (i \neq i[1], i[2], \ldots, i[n])\}$, where $\{i[1], i[2], \ldots, i[n]\}$ is a proper subset of $\{1, 2, \ldots, m\}$.

For any parallelotope P, we use $g(P)$ to denote the barycenter (in the sense of Euclidean geometry) of P, and, for any half open parallelotope P', $g(P')$ shall denote the barycenter of the closure $\overline{P'}$ of P'.

The sum of finitely many parallelotopes is called a *polyparallelotope*, and the sum of finitely many half open parallelotopes is called a *half open polyparallelotope*. In R^m, if the intersection M of an $(m-1)$-dimensional hyperplane and the boundary (in the topological sense) ∂P of a polyparallelotope P contains an $(m-1)$-dimensional rectangle, then the intersection M is called a *face* of P, and each face of the closure $\overline{P'}$ of a half open polyparallelotope P' is called a *face* of P'.

3. Compactification of I_n^{2n+1}

Theorem 3.1. *Let G be a G_δ-set of I^{2n+1} $(\subset R^{2n+1})$ containing I_n^{2n+1} and X be an n-dimensional space. Then G contains a subset which is homeomorphic to X.*

Proof. Let $F=I^{2n+1}-G$. Then F is an F_σ-set in I^{2n+1}. We may write $F=\bigcup_{i=1}^{\infty} F_i$ (each F_i is a closed set of I^{2n+1}). For each integer k ($\geqslant 0$), we define the set of parallelotopes $S(k)$ by induction as follows:

$S(0)=\{I^{2n+1}\}$. Define $S(k+1)$ ($k\geqslant 0$), assuming that $S(k)$ is defined, as follows:

$$S(k)=\{T(k,1),\ T(k,2),\dots,\ T(k,s(k))\}$$

is a set of parallelotopes in R^{2n+1}. Let $T(k,i)$ be an element of $S(k)$, a be a vertex of $T(k,i)$, and $\{j[1], j[2],\dots, j[n+1]\}\subset\{1,2,\dots,2n+1\}$. Let

$$\begin{aligned}&P(k,i,a;j[1],[2],\dots,j[n+1])\\&\quad=\{x\in T(k,i)\,|\,x^{(j[l])}=a^{(j[l])}\ (l=1,2,\dots,n+1)\}.\end{aligned}$$

We write $P(k,i,a;j[1],\dots,j[n+1])$ as P for short. Now assume that $S(k)$ is defined so that every P is contained in I_n^{2n+1}. P does not intersect F_{k+1}. There is a rational number $d(k)$ (>0) such that the parallelotopes

$$\begin{aligned}&T(k,i,a;j[1],\dots,j[n+1])\\&\quad=\{x\in T(k,i)\,|\,a^{(j[l])}-d(k)\leqslant x^{(j[l])}\leqslant a^{(j[l])}+d(k)\ (l=1,2,\dots,n+1)\}\end{aligned}$$

do not intersect F_{k+1} for all combinations $\{i,a,j[1],\dots,j[n+1]\}$ ($i=1,2,\dots,s(k)$; a is some vertex of $T(k,i)$; $\{j[I],\dots,j[n+1]\}\subset\{1,2,\dots,2n+1\}$). Denote these parallelotopes $T(k,i,a;j[1],\dots,j[n+1])$ by $T(k,i)_1$, $T(k,i)_2,\dots,T(k,i)_t$. Let

$$\begin{aligned}&J(k,i;\alpha(1),\alpha(2),\dots,\alpha(p))\\&\quad=\Big\{x\ \Big|\ x\in\bigcap_{q=1}^{p}T(k,i)_{\alpha(q)},\\&\qquad\qquad x\notin T(k,i)_\gamma \text{ for each } \gamma\ (\gamma\neq\alpha(1),\alpha(2),\dots,\alpha(p))\Big\}.\end{aligned}$$

The closure of each $J(k,i;\alpha(1),\dots,\alpha(p))$ is a parallelotope or $\emptyset$. Denote these parallelotopes by $J(k,i;1)$, $J(k,i;2),\dots,J(k,i;h(k))$. Subdivide each $J(k,i;h)$ into 2^{2n+1} parallelotopes by means of $2n+1$ hyperplanes which contain $g(J(k,i;h))$ and are parallel to $2n+1$ coordinate hyperplanes respectively (where $g(J)$ is the barycenter of J). Let $S(k+1)$ be the set of all such parallelotopes produced from all elements of $S(k)$. Let

$$C(k)=\{x\,|\,x\in T \text{ for some element } T \text{ of } S(k)\}\qquad(k=1,2,\dots). \tag{1}$$

Then,

$$C(1)\supset C(2)\supset\cdots,\qquad \operatorname{Mesh} S(k)\to 0\quad(k\to\infty).$$

Now, let X ($\in \mathscr{C}$) be an n-dimensional space. Since $I^{2n+1}-I_n^{2n+1}$ is an n-dimensional universal set, there exists a compactification Y of X such that $\dim X = \dim Y = n$ and $Y \subset I^{2n+1}-I_n^{2n+1}$. We define countably many half open polyparallelotopes $K(k)$ ($k=1,2,\ldots$) as follows:

Let $\{j[1], j[2], \ldots, j[n+1]\} \subset \{1, 2, \ldots, 2n+1\}$ in I^{2n+1}, and let

$$L(j[1], j[2], \ldots, j[n+1])$$
$$= \{x \in I^{2n+1} \mid x^{(j[1])} = x^{(j[2])} = \cdots = x^{(j[n+1])} = \tfrac{1}{2}\}$$

(we denote $L(j[1], \ldots, j[n+1])$ by L for short). It follows that $L \cap Y = \emptyset$. There exists a rational number $r(1)$ such that the half open parallelotopes

$$M(j[1], \ldots, j[n+1])$$
$$= \{x \in I^{2n+1} \mid \tfrac{1}{2} - r(1) < x^{(j[l])} < \tfrac{1}{2} + r(1)\ (l = 1, 2, \ldots, n+1)\}$$

do not intersect Y for all combinations $\{j[1], j[2], \ldots, j[n+1]\}$ ($\subset \{1, 2, \ldots, 2n+1\}$). Let

$$K(1) = \bigcup \{M(j[1], \ldots, j[n+1]) \mid \{j[1], \ldots, j[n+1]\} \subset \{1, 2, \ldots, 2n+1\}\}.$$

Divide $I^{2n+1} - K(1)$ into parallelotopes by means of all hyperplanes (of R^{2n+1}) each of which contains some face of $K(1)$, and let $A(1)$ be the set of all these parallelotopes produced in this way. Subdivide each V ($\in A(1)$) into parallelotopes by means of $2n+1$ hyperplanes which contain $g(V)$ and are parallel to $2n+1$ coordinate hyperplanes respectively. Let $B(1)$ be the set of all parallelotopes produced from all elements of $A(1)$ in this way. Assume that the half open polyparallelotope $K(k)$ is defined and

$$K(k) = \bigcup \{M_\beta \mid \beta = 1, 2, \ldots, \zeta(k)\}$$

where each M_β is a maximal half open parallelotope contained in $K(k)$. We shall define $K(k+1)$ using $K(k)$. Let H be the sum of all hyperplanes (in R^{2n+1}) each of which contains some face of $K(k)$. H divides I^{2n+1} into a finite number u of parallelotopes $N_1, N_2, \ldots, N_u$. For any N_i ($1 \leq i \leq u$), either N_i is contained in the closure $\overline{M_\beta}$ (in R^{2n+1}) of some M_β or N_i does not contain any inner-point of M_β for all β. Let $N_1', N_2', \ldots, N_{u'}'$ be those N_i which do not contain any inner-point of M_β for all β. For each N_s' ($1 \leq s \leq u'$), subdivide N_s' into parallelotopes by means of $2n+1$ hyperplanes which contain $g(N_s')$ and are parallel to $2n+1$ coordinate hyperplanes respectively. Let $B(k) = \{W_1, W_2, \ldots, W_\eta\}$ be the set of all such parallelotopes produced from $N_1', \ldots, N_{u'}'$ in this way. Let $\{j[1], j[2], \ldots, j[n+1]\} \subset \{1, 2, \ldots, 2n+1\}$. For each element W_ξ of $B(k)$, we define

$$L(W_\xi; j[1], \ldots, j[n+1])$$
$$= \{x \in W_\xi \mid x^{(j[l])} = g(W_\xi)^{(j[l])}\ (l = 1, 2, \ldots, n+1)\}$$

(we denote $L(W_\xi; j[1], \ldots, j[n+1])$ by L for short). Then, $L \cap Y = \emptyset$. There exists a rational number $r(k+1)$ such that the half open parallelotopes

$$M(W_\xi; j[1], \ldots, j[n+1])$$
$$= \{x \in W_\xi \mid g(W_\xi)^{(j[l])} - r(k+1) < x^{(j[l])} < g(W_\xi)^{(j[l])} + r(k+1)$$
$$(l = 1, 2, \ldots, n+1)\}$$

do not intersect Y for all combinations $\{W_\xi, j[1], \ldots, j[n+1]\}$ ($W_\xi \in B(k)$, $\{j[1], \ldots, j[n+1]\} \subset \{1, 2, \ldots, 2n+1\}$). Let

$$K(k+1) = \bigcup \{M(W_\xi; j[1], \ldots, j[n+1]) \mid W_\xi \in B(k),$$
$$\{j[1], \ldots, j[n+1]\} \subset \{1, 2, \ldots, 2n+1\}\}.$$

Thus $K(k)$, $B(k)$ $(k = 1, 2, \ldots)$ are defined.

$B(k)$ is similar to $S(k)$, in the sense of classes of (point-) sets, and, setting

$$D(k) = \{x \mid x \in V \text{ for some } V \in B(k)\} \quad (k = 1, 2, \ldots),$$
$$D(1) \supset D(2) \supset \cdots, \qquad \text{Mesh } B(k) \to 0 \quad (k \to \infty).$$

There is a bijection f which maps $\bigcup_{k=1}^{\infty} B(k)$ onto $\bigcup_{k=1}^{\infty} S(k)$ such that

(i) f maps $B(k)$ onto $S(k)$ and the correspondence V $(\in B(k)) \leftrightarrow f(V)$ $(\in S(k))$ is similar in the sense of classes of (point-) sets $(k = 1, 2, \ldots)$,

(ii) if $V \in B(k)$, $V' \in B(k+1)$ and $V \supset V'$, then $f(V) \supset f(V')$.

Therefore, if we set $K = \bigcup_{k=1}^{\infty} K(k)$, then the complement $I^{2n+1} - K$ of K is homeomorphic to C $(= \bigcap_{k=1}^{\infty} C(k))$. Since $I^{2n+1} - K$ contains Y, C has a subset which is homeomorphic to Y. Therefore, from $C \subset G$, G has a subset which is homeomorphic to Y, and therefore has a subset which is homeomorphic to X. $\square$

Theorem 3.2. *Let C $(\in \mathscr{E})$ be any compactification of I_n^{2n+1}. Then, for an arbitrary n-dimensional space X $(\in \mathscr{E})$, C has a subset which is homeomorphic to X.*

Proof. $C \subset R^m$ for some integer m (> 0) because $C \in \mathscr{E}$. We may assume $m \geq 2n+1$. Let

$$I[n] = \{x \in I^m \mid \text{at most } n \text{ elements of } \{x^{(1)}, x^{(2)}, \ldots, x^{(2n+1)}\} \text{ are}$$
$$\text{irrational, } x^{(l)} = 0 \ (l = 2n+2, 2n+3, \ldots, m)\}.$$

Then $I[n]$ is homeomorphic to I_n^{2n+1} and C is a compactification of $I[n]$. Let $I[n]'$ $(\subset C)$ be a set which is homeomorphic to $I[n]$, and let f be a homeomorphic mapping of $I[n]$ onto $I[n]'$. There exist two G_δ-sets G, G' (of R^m) and a homeomorphic mapping $f': G \to G'$ such that G and G' contain $I[n]$ and $I[n]'$, respectively, and f' is an extension of f [5]. $(f')^{-1}(G' \cap C)$ is a G_δ-set (of R^m) containing $I[n]$. Let

$$E[n] = ((f')^{-1}(G' \cap C)) \cap \{x \in R^m \mid 0 \leq x^{(l)} \leq 1 \ (l = 1, 2, \ldots, 2n+1),$$
$$x^{(l)} = 0 \ (l = 2n+2, 2n+3, \ldots, m)\},$$

then $E[n]$ is a G_δ-set (of R^m) containing $I[n]$. If we set

$$E[n]''=\{y\in R^{2n+1}\mid \exists x\ (\in E[n])\ y^{(l)}=x^{(l)}\ (l=1,2,\ldots,2n+1)\},$$

then $E[n]''$ is a G_δ-set (of R^{2n+1}) containing I_n^{2n+1}, and, by Theorem 3.1, $E[n]''$ has a subset $X[n]''$ which is homeomorphic to X. Let

$$X[n]=\{x\in R^m\mid \exists y\ (\in X[n]'')\ x^{(l)}=y^{(l)}\ (l=1,2,\ldots,2n+1),$$
$$x^{(l)}=0\ (l=2n+2,2n+3,\ldots,m)\}.$$

Then, $X[n]\subset E[n]$. Therefore

$$f'(X[n])\subset C.$$

Therefore, since X is homeomorphic to $X[n]$, X can be embedded in C. □

4. $d(X)\leqslant \dim X$

Theorem 4.1. *For any X ($\in \mathscr{E}$),*

$$d(X)\leqslant \dim X$$

(*where d is an arbitrary function satisfying the conditions* (A1)–(A5) *in Section* 1 *for* $\mathscr{Q}=\mathscr{E}$).

Proof. It is obvious if $\dim X=-1$.

Let $\dim X=n$ $(0\leqslant n<\infty)$.

In R^{2n+1}, I_n^{2n+1} is the sum of countably many n-parallelotopes. Since $d(D)=n$ for any n-dimensional interval D from the conditions (A3) and (A5), we have $d(I_n^{2n+1})=n$ by condition (A2). By condition (A4), there exists a compactification A ($\in \mathscr{E}$) of I_n^{2n+1} such that $d(A)=d(I_n^{2n+1})$. By Theorem 3.2, A has a subset X' which is homeomorphic to X. Therefore we have

$$d(X)=d(X')\leqslant d(A)$$

by conditions (A3) and (A1). Since

$$d(A)=d(I_n^{2n+1})=n,$$

we have $d(X)\leqslant n$. And $d(X)\leqslant \dim X$ since $\dim X=n$. □

5. $d(X)=\dim X$

The proof of the following lemma is essentially due to Menger [7, p. 208].

Lemma 5.1. *If X ($\in \mathscr{E}$) has* $\dim X\geqslant 1$ *and $\mathscr{P}$ ($\subset \mathscr{E}$) satisfies the following conditions* (A1′)–(A5′):

(A1′) *if $Y'\subset Y$ and $Y\in \mathscr{P}$, then $Y'\in \mathscr{P}$,*

(A2′) *if Y is the sum of countably many closed sets ($\in \mathcal{P}$) of Y, then $Y \in \mathcal{P}$,*
(A3′) *if $Y \in \mathcal{P}$ and Y is homeomorphic to Y', then $Y' \in \mathcal{P}$,*
(A4′) *if $Y \in \mathcal{P}$, then there exists a compactification Y' of Y such that $Y' \in \mathcal{P}$,*
(A5′) *$X \in \mathcal{P}$,*
then $\mathcal{P}$ contains a (1-dimensional) arc as an element.

Proof. (1) There exists a compactification X_0 ($\in \mathcal{P}$) of X, and there exists a connected component X' of X_0 containing at least two points. For some integer m (>0), $X' \subset I^m$ ($\subset R^m$). Define the sets of m-cubes $Q[n]$ ($n = 0, 1, 2, \ldots$) as follows:

Let $Q[0] = \{I^m\}$.

Divide I^m into 2^{mn} m-cubes each of which has edges of length $(\frac{1}{2})^n$, and let $Q[n]$ be the set of these 2^{mn} m-cubes.

For each element $Q(n, i)$ ($i = 1, 2, \ldots, mn$) of $Q[n]$ ($n = 0, 1, 2, \ldots$), there exists a set $X(n, i)$ ($\subset Q(n, i)$) such that $X(n, i)$ contains all vertices of $Q(n, i)$ and is homeomorphic to X'. Let

$$Y = \bigcup \{X(n, i) \mid n = 0, 1, 2, \ldots; i = 1, 2, \ldots, mn\}.$$

Then $Y \in \mathcal{P}$ since each $X(n, i)$ is compact and is in $\mathcal{P}$. Obviously Y is connected and dense in I^m.

(2) Let S be an arbitrary set with $Y \subset S \subset I^m$. Then S is connected. We shall prove that S is locally connected.

Let p be an arbitrary point of S and $U(p; \varepsilon)$ be the ε-neighborhood (in R^m) of p, where ε is an arbitrary real number (>0). Let a, b ($a \neq b$) be two arbitrary points of $U(p; \varepsilon) \cap S$. For some integer n, there exist finitely many elements $Q(n, i_1)$, $Q(n, i_2), \ldots, Q(n, i_j)$ of $Q[n]$ such that

$$a \in Q(n, i_1), \qquad b \in Q(n, i_j),$$

$$Q(n, i_k) \subset U(p; \varepsilon) \quad (k = 1, 2, \ldots, j),$$

and

$$Q(n, i_k) \cap Q(n, i_{k+1}) \neq \emptyset \quad (k = 1, 2, \ldots, j-1).$$

Let $P = \bigcup_{k=1}^{j} Q(n, i_k)$ and let

$$T = \bigcup \{X(n', i') \mid X(n', i') \subset P\}.$$

Then, T is connected and the closure of T is equal to P. Therefore $T \cup \{a, b\}$ is connected, since $T \subset T \cup \{a, b\} \subset P$. It follows that $U(p; \varepsilon) \cap S$ is connected. Thus S is locally connected.

(3) There exists a compactification Z ($\in \mathcal{P}$) of Y. We may assume that $Z \subset R^k$ ($k \geq m$) for some integer k. Let Y' be the image of Y embedded in Z and f the embedding mapping $Y \to Z$. There exist two G_δ-sets G, G' (of R^k) and a homeomorphic mapping $f' : G \to G'$ such that $Y \subset G$, $Y' \subset G'$ and f' is an extension of f. Let

$$D = ((f')^{-1}(G' \cap Z)) \cap I^m \; (\subset R^k)$$

and let $D'=f'(D)$. Then both D and D' are G_δ-sets (of R^k). Since $Z\in\mathcal{P}$, we have $D'\in\mathcal{P}$, and therefore $D\in\mathcal{P}$. Since $Y\subset D\subset I^m$, D is connected and locally connected by (2). Since D is a G_δ in R^k, there exists a complete metric space C which is homeomorphic to D [2]. Since C is connected and locally connected, C has a (1-dimensional) arc as a subset. Therefore D has an arc as a subset, and it follows that $\mathcal{P}$ has an arc as an element. □

Lemma 5.2. *If* X $(\in\mathcal{E})$ *has* $\dim X\geq 1$, *then* $d(X)\geq 1$ (*where d is an arbitrary set function satisfying the conditions* (A1)–(A5) *for* $\mathcal{Q}=\mathcal{E}$).

Proof. Let $d(X)=t$. Since the class $\mathcal{P}$ consisting of all X' $(\in\mathcal{E})$ with $d(X')\leq t$ satisfies the conditions (A1′)–(A5′) in Lemma 5.1, $\mathcal{P}$ contains an arc L as an element. Since $d(L)=1$, we have $t\geq 1$. Therefore $d(X)\geq 1$. □

Theorem 5.3. *For arbitrary* X $(\in\mathcal{E})$,

$$d(X)\geq\dim X$$

(*where d is an arbitrary set function satisfying the conditions* (A1)–(A6) *for* $\mathcal{Q}=\mathcal{E}$).

Proof. (1) If $d(X)=-1$, then $\dim X=-1$, therefore $d(X)\geq\dim X$.

(2) Let $-1<d(X)\leq 0$. Since X is not empty, X has at least one point. Hence $d(X)\geq 0$, and therefore $d(X)=0$. If we suppose $\dim X\geq 1$, then, by Lemma 5.2, we have $d(X)\geq 1$, and this contradicts the assumption $d(X)\leq 0$. Therefore $d(X)\geq\dim X$.

(3) Let $d(X)=t$ $(0\leq n-1\leq t<n$ where n is an integer). By (A6), there exist n subsets $X_1, X_2,\ldots,X_n$ of X such that

$$X=X_1\cup X_2\cup\cdots\cup X_n,\qquad d(X_i)\leq 0\quad(i=1,2,\ldots,n).$$

By (2), $\dim X_i\leq 0$ $(i=1,2,\ldots,n)$. Therefore

$$\dim X=\dim\left(\bigcup_{i=1}^{n}X_i\right)\leq n-1.$$

It follows that $d(X)\geq\dim X$. □

From Theorem 4.1 and Theorem 5.3, we have the following theorem:

Theorem 5.4. *For arbitrary* X $(\in\mathcal{E})$,

$$d(X)=\dim X$$

(*where d is an arbitrary set function satisfying the conditions* (A1)–(A6) *for* $\mathcal{Q}=\mathcal{E}$).

6. Independence of the axioms

We shall prove that the conditions (A1)–(A6) mentioned in Section 1 are independent for the case $\mathcal{Q}=\mathcal{E}$.

• *Independence of* (A1).

Let $\mathcal{N}$ be the class of sets N such that N has dim $= 1$ and is a non-Borel subset of a Euclidean space. If X ($\in \mathscr{E}$) has dim $= 1$ and satisfies the following condition $\triangle$:

$\triangle$: for any countable closed covering $\mathscr{C} = \{X_i \mid i = 1, 2, \ldots\}$ of X, there exists a member X_i of $\mathscr{C}$ such that $X_i \in \mathcal{N}$, then let $d(X) = 2$. And, for the other X ($\in \mathscr{E}$), let $d(X) = \dim X$.

(1) There exists X ($\in \mathscr{E}$) having dim $= 1$ and $d(X) = 2$. (For example, $X = N \times I$ ($\subset R^2$) where N is a non-Borel set of I.) There exists a compactification X' of X such that $\dim X' = \dim X = 1$. Since X' is a Borel set in a Euclidean space and $\dim X' = 1$, we have $d(X') = 1$. As X is embedded in X', this function d does not satisfy (A1).

(2) Let $X \in \mathscr{E}$, $X = \bigcup_{i=1}^{\infty} X_i$ and $d(X_i) \leq 1$ ($i = 1, 2, \ldots$), where each X_i is a closed set of X. We shall prove $d(X) \leq 1$. Since $d(X_i) \leq 1$, for any countable closed covering $\mathscr{C}_i = \{X_{i,j} \mid j = 1, 2, \ldots\}$ of X_i, each $X_{i,j}$ is either a non-Borel set with dim $= 0$ or a Borel set with dim ≤ 1. Since each $X_{i,j}$ is closed in X, from the definition, $d(X) \leq 1$.

It is easily established that this function d satisfies (A2) for the other cases and also satisfies (A3)-(A6).

• *Independence of* (A2).

If X ($\in \mathscr{E}$) is homeomorphic to a subset of the 1-dimensional interval I, then let $d(X) = \dim X$. If X ($\in \mathscr{E}$) has $\dim X = 1$ and X is not homeomorphic to any subset of I, then let $d(X) = 2$. And, for each space X ($\in \mathscr{E}$) having $\dim X \geq 2$, let $d(X) = \dim X$.

• *Independence of* (A3).

Let $\mathscr{B}$ be the class of sets B such that B has dim $= 1$ and is the image of a second category set of R^2 by an isometric mapping. If X ($\in \mathscr{E}$) has dim $= 1$ and has a metric subspace which is a member of $\mathscr{B}$, then let $d(X) = 2$. And, for the other X ($\in \mathscr{E}$), let $d(X) = \dim X$.

Let B be a second category subset of R^1 with $\dim B = 0$, and C be a nowhere dense subset of R^1 which is homeomorphic to B. Then, $B \times I$ is homeomorphic to $C \times I$. However, $d(B \times I) = 2$, $d(C \times I) = 1$. Therefore, this function d does not satisfy (A3). It is easily established that d satisfies (A1), (A2), (A4)-(A6).

• *Independence of* (A4).

Let $\mathscr{B}$ be the class of sets having the form $\{B \mid B$ is the sum of countably many closed subsets $B_1, B_2, \ldots$ such that each B_i is homeomorphic to a subset of the interval $I\}$. If X ($\in \mathscr{E}$) is a member of $\mathscr{B}$ and has $\dim X = 1$, then let $d(X) = 1$. If X ($\in \mathscr{E}$) is not a member of $\mathscr{B}$ and has $\dim X = 1$, then let $d(X) = 2$. And, for the other X ($\in \mathscr{E}$), let $d(X) = \dim X$.

It is easily established that d satisfies (A1)-(A3), (A5), (A6). We shall prove that d does not satisfy (A4). Denote points of R^2 by (x, y) where x and y are real numbers. Let

$$C = \{(x, y) \in R^2 \mid x \text{ is irrational}, 0 < x < 1, 0 \leq y \leq 1\}.$$

Then C has dim = 1. But $d(C) = 2$, because, if $C = \bigcup_{i=1}^{\infty} C_i$ where each C_i is closed in C, then some C_i contains uncountably many pairwise disjoint segments. Therefore $C_i \not\subset I$, hence $C \notin \mathscr{B}$. On the other hand, $d(I_1^3) = 1$, since I_1^3 is the sum of countably many segments. By Theorem 3.2, any compactification A of I_1^3 contains a subset which is homeomorphic to C. Therefore $d(X) \geq 2$. Therefore, there is no compactification A of I_1^3 such that $d(I_1^3) = d(A)$. Hence, this function d does not satisfy (A4).

- *Independence of* (A5).

Let $d(\emptyset) = -1$, and, let $d(X) = 0$ for every X ($\in \mathscr{C}$) which is not empty.

- *Independence of* (A6).

Let $Z(p)$ be the cyclic group of order p, where p is a prime number. If we let $d(X) = \text{c-dim}_{Z(p)} X$ where $\text{c-dim}_{Z(p)} X$ is the cohomological dimension of X ($\in \mathscr{C}$), then $d(X)$ satisfies the conditions (A1)-(A5) [4] but does not satisfy the condition (A6). (Because, if we suppose that $d(X)$ satisfies the conditions (A1)-(A6), then, by Theorem 5.4, we have $d(X) = \dim X$ for every X ($\in \mathscr{C}$), and is contradictory to $\text{c-dim}_{Z(p)} X \neq \dim X$ for some X.)

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